Mathematics often is considered the most abstract of all knowledge, yet it is rooted in the soil of practical problems. The earliest recorded use of math is by accountants in ancient Sumeria. They tracked the amount of crops and livestock owned by farmers and taxed by kings. In the modern, high-tech environment, some of the most challenging problems retain this age-old but practical flavor.

What is the shortest route to visit 50 cities?

What is the best way to load 45 oddly shaped boxes onto four trucks?

How do you build an inexpensive communications network with enough redundancy to avoid collapse when a fiber-optic line is clipped by a backhoe?

Tom Trotter and Hal Kierstead spend their time thinking about these types of problems. The problems are easy to understand, but very difficult to solve.

Trotter and Kierstead are experts in a field of mathematics called combinatorics, or discrete math. Professors of mathematics at Arizona State University, they study the properties of discrete structures such as maps and networks, and the nature of relationships among them.

“A painful lesson that one learns in discrete mathematics is the subtle distinction between what’s hard and what’s easy. Just a little twist in a simple problem, and it suddenly becomes terribly difficult,” says Trotter, a Regents’ Professor of Mathematics.
Trotter admits that this surprising feature is what first attracted him to study discrete math some 25 years ago. Trotter’s career path has also taken some surprising twists. He worked for the National Aeronautical and Space Administration as an engineer before attending graduate school at the University of Alabama on a NASA fellowship.

He entered graduate school fully intending to return to the space industry. But he switched fields and completed a doctorate in topology in 1969. That fall, he took a faculty position at The Citadel and taught undergraduate mathematics. He even coached the swimming team for a season. It was not enough.

“I loved teaching mathematics. But I realized that I was dying intellectually—I lacked the day to day challenge of research to reinforce and refresh my teaching,” Trotter recalls.

In 1971, Trotter attended a National Science Foundation-sponsored mathematics conference on combinatorics at Maine’s Bowdoin College. He discovered he had a knack for this new subject.

“I seemed to have insights into what should be true as well as some reasonable guesses as how one should go about proving things,” he says. “I was hooked. I dropped topology like a rock and dived headfirst into combinatorics.”

Trotter’s excitement for the subject continues to this day. Combinatorics frequently requires high-powered computer resources. Trotter explains that up until the late 1970s, an acceptable mathematical proof was one that a single person could follow line by line to conclusion. However, recent important proofs have depended on computer assistance.

Such hybrid proofs are intrinsically long; no amount of mathematical cleverness can apparently get around the need for pages upon pages of particular cases necessary to the overall proof. Trotter and Kierstead are untroubled by such proofs.

“Within our field, computers are now used in proving and discovering theorems. Some researchers write highly creative codes (computer programs) to implement algorithms,” Trotter explains. “Others use sophisticated packages to explore and test new ideas. Computers don’t replace paper and pencil, nor have they in any way diminished the emphasis on clear thinking and reasoning. They add a new dimension to how we work.”

To a mathematician, a proof is a logical demonstration that a mathematical hypothesis or conjecture is true. If the proof is judged sufficiently rigorous, the hypothesis becomes a theorem. A proof is deemed rigorous if it starts from premises which are accepted and proceeds to conclusion in logical steps small enough that anyone with training can follow each step without contradiction.

Oddly enough, there is a social component to proof. Some proofs, when first published, are not accepted by knowledgeable experts in the field. Mathematicians know that mistakes occur, even in published work. In-depth review of the most difficult work can take months, even years.

During this time, experts dissect the submitted proof. Over time, its correctness is confirmed or a fatal flaw is identified. “Proof is a social construct,” Trotter says, “but it is a construct based on thousands of years of mathematical and logical practice and experience.” Successful mathematicians may approach their work in quite different ways.

“Hal tends to be precise,” says Trotter of colleague Kierstead. “His background is in mathematical logic. He provides details for ideas as he develops them. As a consequence, he rarely makes mistakes.”

Trotter describes his own style as more informal. “I often rush ahead to where an idea might lead, and I make mistakes all the time. Maybe that’s why we make such a good team.”
The Nodes and Links of Problem Solving

Tom Trotter and Hal Kierstead study graphs, but not the kind of graphs you used to make in chemistry or business class. To a discrete mathematician, a graph is a set of points, or “nodes,” and a set of edges, or “links,” between certain pairs of points.

For example, consider a telecommunications network as a graph, with nodes representing switching facilities and links representing fiber optic lines. Graphs occur many places in everyday life: consider a fisherman’s net, a spider web, an organizational chart, a job schedule, even an Arizona road map.

The kind of problem the graph represents depends on the properties assigned to the nodes and links. Small variations in properties can translate into large variations in the difficulty and applicability of particular problems. Because of this, problems in discrete math that seem unrelated frequently have deep, underlying structural similarities.

Think about distance tables found on the back of road maps. The tables list the shortest distance between certain pairs of cities. The table represents a problem that is easy to solve. Using the calculating power of a small personal computer, anyone could easily prepare such a table for the 10,000 largest cities in America.

Now comes the hard problem. Is there a route through the graph that will allow the now-famous traveling salesperson to tour 1,000 nodes (cities) but only visit each node exactly once? Trotter says that no computer yet exists that can factor 200 digit numbers—at least not all 200 digit numbers. The difficulty of factoring is the basis for an important technique called public key encryption, a method for secure data transmission.

The fundamental difficulty in both problems is that the number of possibilities is extraordinarily large. The complications that come with size are not artificial. Large problems occur quite naturally in the real world.

Optimization problems with more than 1,000 variables occur routinely in economic modeling, marketing, weather forecasting, telecommunications networks, microchip design, airline operations, manufacturing, and city planning.

To understand the critical role played by the size of a problem, Trotter suggests considering the challenge of determining whether there is a tour visiting all vertices in a graph exactly once. If all possible pairs of nodes are linked by an edge, then the answer is yes. But only certain pairs are linked.

One possible approach is to use a high speed computer to list all possible sequences of vertices, then test them one by one to see if each consecutive pair of nodes is linked by an edge.

If the graph has 1,000 vertices, then the number of sequences to test is the product 1,000 x 999 x 998 x ... etc. x 2 x 1. If the computer needs one second to test each possibility, then it would take about 10^2500 years to test them all. That is 10 followed by 2,500 zeroes. Quite a bit of time, indeed.

Consider that an incredible breakthrough in chip design allows us to test 10^10 possibilities per second. The result? You can knock off the last 10 zeroes in the time estimate. Time necessary has dropped in years to 10 followed by 2,490 zeroes. Regardless of chip speed, no one will be around to see the end result.

“But if you can come within a reasonable degree of the right answer, in a reasonable amount of time, and at reasonable cost, you may consider that you’ve solved the problem.”

Hal Kierstead

Trotter says that by asking a slightly different question, one gets a different problem. For example, when building a telephone or other communications network, how do you connect all the nodes at the least possible cost without doubling up?

In contrast to the Traveling Salesperson Problem, which for large numbers of nodes seems unsolvable, this one is easy. Simply make the cheapest or shortest connections possible until every node has a link. There are no redundancies.

However, give the circuit some degree of redundancy—some backups or failsafe—and the problem quickly turns back into a Traveling Salesperson-type Problem. A complete circuit (called a Hamiltonian circuit) would provide the most economical way to make sure every node had exactly two connections.

Another tweak of graph properties creates a fundamentally different problem: How to color a map?

Each node represents a country or state. The links represent borders between countries. Without assigning nodes or links any weights or costs or distances, the problem becomes how to color the map without assigning two adjoining states the same color.

First posed by a British graduate student in 1852, this classic problem was based on the supposition that four colors were all that would be needed.

The hypothesis was proved in 1976. What shocked many examiners was that this proof of a seemingly simple problem ran to hundreds of pages and required 1,200 hours of computer time to solve.

Variations on map coloring problems can be even more daunting. Imagine, for example, extending the coloring problem to solids in space instead of areas on a plane; or, as in Kierstead’s work, to adversarial situations.
Add a second player with his own set of crayons who may be incompetent, or worse, antagonistic, and the problem becomes even more difficult.

Kierstead and Trotter have proved that when the colorers take turns, 33 colors are enough to color a planar graph when alternating turns with a partner who may in fact be uncooperative. It is not known whether 33 is the best possible answer, although at least 8 colors are required.

Individually and as a team, Kierstead and Trotter have the best-known results on a large number of optimization problems. The problems span topics in dynamic storage allocation, on-line algorithms, discrete geometry, and others involving graphs and partially ordered sets.

Trotter says that map and graph coloring problems are classical optimization problems. They occur in the real world as resource allocation problems.

Nodes linked by an edge represent conflicting states that must be assigned distinct resources. Examples include the need to separate television transmitters located in close proximity with different channel assignments, storing chemicals with dangerous interactions in separate rooms, or assigning concurrent processes disjoint blocks of memory in a computer.

Coloring problems are structurally similar to bin-packing problems. Described simply, such problems center on how best to fit items into a given space, be it a bin, a truck, or a warehouse. An example arises in the clothing industry when a company wants to cut the maximum pairs of jeans from a single bolt of cloth.

If all the items to be packed are known, and you have a fast computer, it may be possible to calculate the best packing solution for that number of items. However, if you are given more to pack each day—also called dynamic situation—the problem becomes much more difficult. Chemical storage and memory allocation are just two dynamic bin-packing problems.

The underlying deep structure of discrete math problems is a recurring theme. Trotter says many of the problems appear to be unsolvable, at least if we insist on finding the precise optimal answer. The explanation for the difficulty is found in ideas from computational complexity, an area of discrete mathematics that provides the mathematical foundations for computer science.

-Tom Trotter

The Arizona State University (ASU) researcher makes extensive use of E-mail and the telephone to work out problems with colleagues. Borrowing a phrase from legendary Hungarian mathematician Paul Erdos, Trotter’s conversations begin with a question, “Is your brain open?” He then launches into a technical discussion of a late night idea, which often blossoms into a daytime breakthrough.

Certain problems in combinatorics have boundaries.

“I like to use the Traveling Salesperson Problem because it such a good example of what discrete mathematics is all about,” Trotter explains. “The problem is easy enough to understand. But consider the technical difficulties in finding the shortest route for visiting 10,000 cities and you gain an appreciation for the special challenges awaiting researchers in this field.”

Getting a precise answer to this optimization problem is not easy. Currently, the world’s record for the Traveling Salesperson Problem is just over 3,000 cities. That solution required the creative energies of four top researchers and the dedicated use of almost one month’s time on a supercomputer.

In sharp contrast, you can get a pretty good approximation to the optimal solution in a very short time. This is a common theme of many discrete math problems.

Typically, a discrete mathematician is expected to give not just a solution, but to also provide a certificate, an proof showing that the answer is in fact reasonable or about as good as can be hoped for.

“In many instances, you know you’re not going to be able to find a usable algorithm for the true optimal solution,” Kierstead says. “But if you can come within a reasonable degree of the right answer, in a reasonable amount of time, and at reasonable cost, you may consider that you’ve solved the problem—at least in the real world.”

Trotter provides another example. “When I was working in industry, my research group was asked to solve a routing problem for one of the Baby Bells. The particular problem involved routing service vehicles, an area where achieving a 10 percent reduction in estimated costs on an annual basis can be of great importance to firms like U.S. West.

“Expenditures on service operations, in terms of equipment and personnel, are staggering, and service is vitally important to their business,” he continues. “In this instance, the technical difficulty of the problem hinged on the issue as to whether or not drivers are allowed to make U-turns in rural areas. Allowing for this possibility, we devised efficient, accurate algorithms, and our corporate clients were very happy. We elected not to tell them that without the U-turns, neither we nor anybody else on the planet could help them.”

Trotter and Kierstead live and breathe their work. Applications of combinatorics are everywhere.

It is now possible, for example, to digitally time-stamp an electronic document to prove that it was completed or sent at a particular time. It also is now possible to safely encode the details of financial transactions so that sensitive data can be transmitted across the Internet. And it is now possible to create automated inventory control systems that are quite accurate even when mistakes are made frequently. All of these advances are based on research in discrete math and the work of mathematicians such as Trotter and Kierstead.

Plenty more problems await solutions.

Combinatorics and discrete mathematics research is supported by the National Science Foundation. For more information, contact William T. Trotter, Ph.D, Mathematics Department, College of Liberal Arts and Sciences, 602.965.5893, or E-mail at trotter@asu.edu