

Sets, Venn Diagrams & Counting

Terri Miller

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SETS

What is a set?

A **set** is a collection of objects. The objects in the set are called elements of the set. A set is **well-defined** if there is a way to determine if an object belongs to the set or not. To indicate that we are considering a set, the objects (or the description) are put inside a pair of set braces, $\{ \}$.

Example 1. Are the following sets well-defined?

- (1) The set of all groups of size three that can be selected from the members of this class.
- (2) The set of all books written by John Grisham.
- (3) The set of great rap artists.
- (4) The best fruits.
- (5) The 10 top-selling recording artists of 2007.

Solution:

- (1) You can determine if a group has three people and whether or not those people are members of this class so this is well-defined.
- (2) You can determine whether a book was written by John Grisham or not so this is also a well-defined set.
- (3) A rap artist being great is a matter of opinion so there is no way to tell if a particular rap artist is in this collection, this is not well-defined.
- (4) Similar to the previous set, best is an opinion, so this set is not well-defined.
- (5) This is well-defined, the top selling recording artists of any particular year are a matter of record.

Equality?

Two sets are equal if they contain exactly the same elements.

Example 2.

- (1) $\{1, 3, 4, 5\}$ is equal to the set $\{5, 1, 4, 3\}$
- (2) The set containing the letters of the word *railed* is equal to the set containing the letters of the word *redial*.

There are two basic ways to describe a set. The first is by giving a description, as we did in Example 1 and the second is by listing the elements as we did in Example 2 (1).

Notation.: We usually use an upper case letter to represent a set and a lower case x to represent a generic element of a set. The symbol \in is used to replace the words “is an element of”; the expression $x \in A$ would be read as x is an element of A . If two sets are equal, we use the usual equal sign: $A = B$.

Example 3. $A = \{1, 2, 3, 5\}$

$B = \{m, o, a, n\}$

$C = \{x | x \geq 3 \text{ and } x \in \mathbb{R}\}$

$D = \{\text{persons} \mid \text{the person is a registered Democrat}\}$

$U = \{\text{countries} \mid \text{the country is a member of the United Nations}\}$

Universe?

In order to work with sets we need to define a **Universal Set**, U , which contains all possible elements of *any* set we wish to consider. The Universal Set is often obvious from context but on occasion needs to be explicitly stated.

For example, if we are counting objects, the Universal Set would be whole numbers. If we are spelling words, the Universal Set would be letters of the alphabet. If we are considering students enrolled in ASU math classes this semester, the Universal Set could be all ASU students enrolled this semester or it could be all ASU students enrolled from 2000 to 2005. In this last case, the Universal Set is not so obvious and should be clearly stated.

Empty?

On occasion it may turn out that a set has no elements, the set is empty. Such a set is called the **empty set** and the notation for the empty set is either the symbol \emptyset or a set of braces alone, $\{\}$.

Example 4. Suppose A is the set of all integers greater than 3 and less than -1 . What are the elements of A ? There are no numbers that meet this condition, so $A = \emptyset$.

Subset.

A is a **subset** of B if every element that is in A is also in B . The notation for A is a subset of B is $A \subseteq B$. Note: A and B can be equal.

Example 5.

$$A = \{0, 1, 2, 3, 4, 5\}$$

$$B = \{1, 3, 4\}$$

$$C = \{6, 4, 3, 1\}$$

$$D = \{0, 1, 2, 5, 3, 4\}$$

$$E = \{\}$$

Which of the sets B, C, D, E are subsets of A ?

$B \subseteq A$ since its elements 1, 3, and 4 are all also in A . C is NOT a subset of A ($C \not\subseteq A$) since there is a 6 in C and there is no 6 in A . D is a subset of A since everything that is in D is also in A ; in fact $D = A$. Finally, E is a subset of A ; this is true since any element that IS in E is also in A .

Notice that every set is a subset of itself and the empty set is a subset of every set. If $A \subseteq B$ and $A \neq B$, then we say that A is a **proper subset** of B . The notation is only a bit different: $A \subset B$. Note the lack of the “equal” part of the symbol.

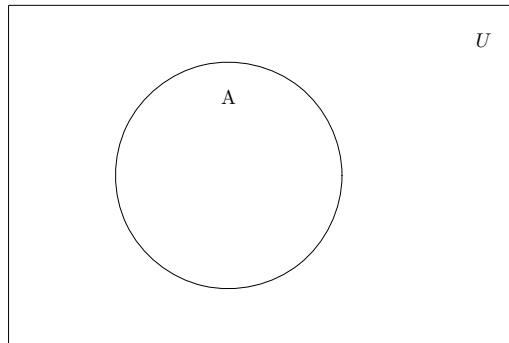
Complement. Every set is a subset of some universal set. If $A \subseteq U$ then the **complement** of A is the set of all elements in U that are NOT in A . This is denoted: A' . Note that $(A')' = A$, i.e. the complement of the complement is the original set.

Example 6. Consider the same sets as in Example 5. It appears that the set of all integers, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, would be a natural choice for the universe in this case. So, we would have

$$\begin{aligned} A' &= \{6, 7, 8, 9\} \\ B' &= \{0, 2, 5, 6, 7, 8, 9\} \\ C' &= \{0, 2, 5, 7, 8, 9\} \\ E' &= U \end{aligned}$$

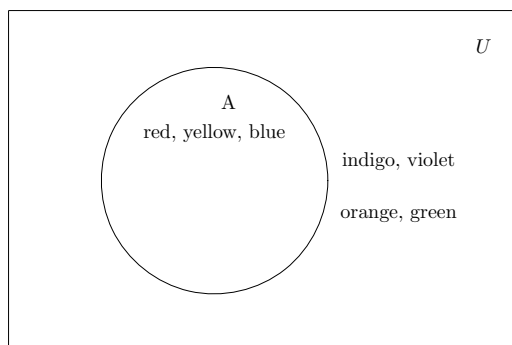
VENN DIAGRAMS

Pictures are your friends! It is often easier to understand relationships if we have something visual. For sets we use Venn diagrams. A **Venn diagram** is a drawing in which there is a rectangle to represent the universe and closed figures (usually circles) inside the rectangle to represent sets.



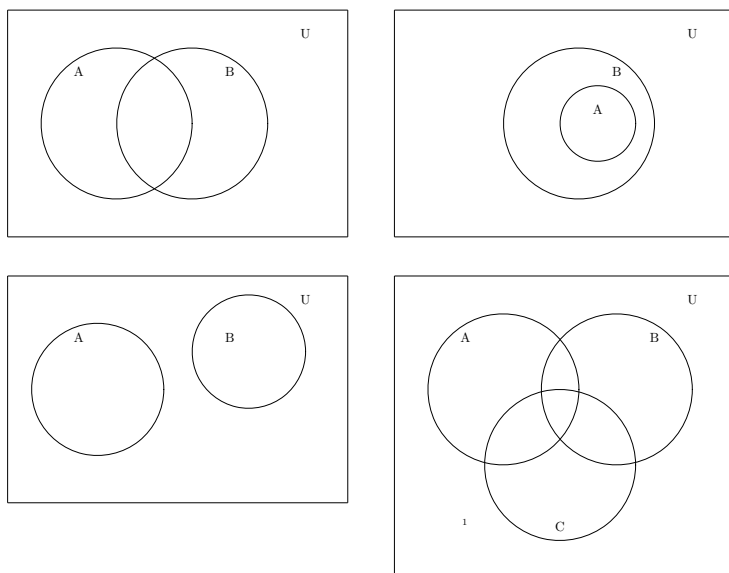
One way to use the diagram is to place the elements in the diagram. To do this, we write/draw those items that are in the set inside of the circle. Those items in the universe that are not in the set go outside of the circle.

Example 7. With the sets $U = \{\text{red, orange, yellow, green, blue, indigo, violet}\}$ and $A = \{\text{red, yellow, blue}\}$, the diagram looks like



Notice that you can see A' as well. Everything in the universe not in A , $A' = \{\text{orange, indigo, violet, green}\}$.

If there are more sets, there are more circles, some pictures with more sets follow.



Set Operations. We will need to be able to do some basic operations with sets.

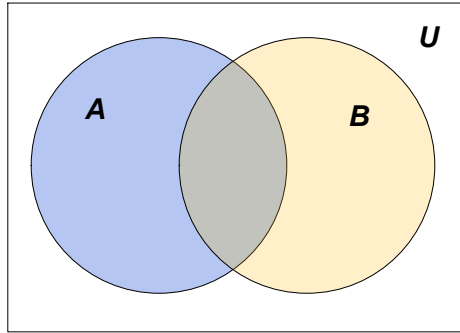
The first operation we will consider is called the union of sets. This is the set that we get when we combine the elements of two sets. The *union* of two sets, A and B is the set containing all elements of both A and B ; the notation for A union B is $A \cup B$. So if x is an element of A or of B or of both, then x is an element of $A \cup B$.

Example 8. For the sets $A = \{\text{bear, camel, horse, dog, cat}\}$ and $B = \{\text{lion, elephant, horse, dog}\}$, we would get $A \cup B = \{\text{bear, camel, horse, dog, cat, lion, elephant}\}$.

To see this using a Venn diagram, we would give each set a color. Then $A \cup B$ would be anything in the diagram with any color.

Note: $A \cup A' = U$, the union of a set with its complement gives the universal set.

Example 9. If we color the set A with blue and the set B with orange, we see the set $A \cup B$ as the parts of the diagram that have any color (blue, beige, orange).

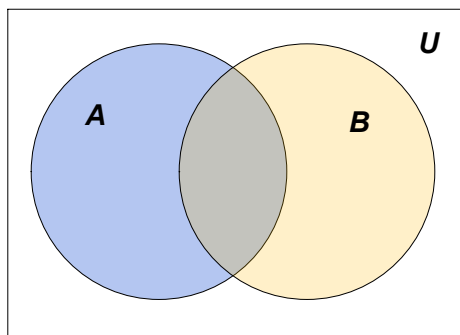


The next operation that we will consider is called the intersection of sets. This is the set that we get when we look at elements that the two sets have in common. The *intersection* of two sets, A and B is the set containing all elements that are in *both* A and B ; the notation for A intersect B is $A \cap B$. So, if X is an element of A and x is an element of B , then x is an element of $A \cap B$.

Example 10. For the sets $A = \{bear, camel, horse, dog, cat\}$ and $B = \{lion, elephant, horse, dog\}$, we would get $A \cap B = \{horse, dog\}$.

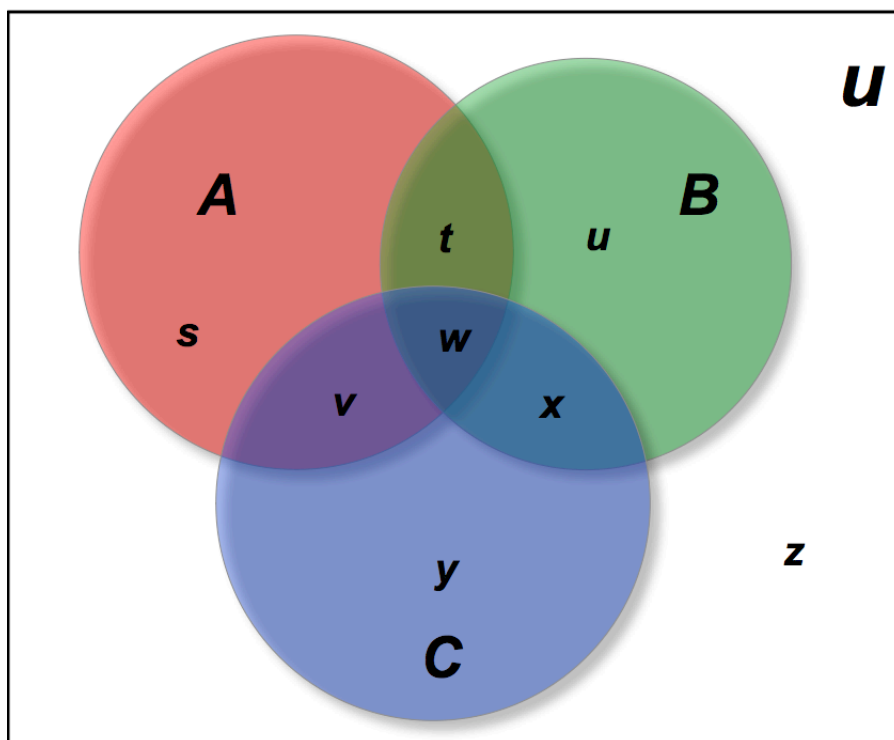
To see this using a Venn diagram, we would give each set a color. Then $A \cap B$ would be anything in the diagram with both colors.

Example 11. If we color the set A with blue and the set B with orange, we see the set $A \cap B$ as the part of the diagram that has both blue and orange resulting in a beige colored "football" shape.



Note: $A \cap A' = \{\}$, the intersection of a set with its complement is the empty set.

We can extend these ideas to more than two sets. With three sets, the Venn diagram would look like



In this diagram, the three sets create several “pieces” when they intersect. I have given each piece a lower case letter while the three sets are labelled with the upper case letters A , B , and C . I will describe each of the pieces in terms of the sets A , B , and C .

With three sets, the keys to completing the venn diagrams are the “triangle” pieces, t , v , w , and x . The darkest blue piece in the center, w , is the intersection of all three sets, so it is $A \cap B \cap C$; that is the elements in common to all three sets, A and B and C . The yellow piece t is part of the intersection of 2 of the sets, it is the elements that are in both A and B but *not* in C , so it is $A \cap B \cap C'$. Similarly v , the purple piece, is the elements that are in both A and C but not in B , $A \cap C \cap B'$. Finally, the blue piece x is the elements that are in both B and C but not in A , $A' \cap B \cap C$.

The next pieces to consider are the football shaped pieces formed by joining two of the triangle pieces. The piece composed of t and w is the elements in both A and B , so it is the set $A \cap B$. The football formed by v and w is the elements that are in both A and C , that is $A \cap C$. The last football, formed by x and w , is the elements in both B and C , i.e. $B \cap C$.

The areas to consider are the large outer pieces of each circle. The red region marked with s is the elements that are in A but not in B or C , this is the set $A \cap (B \cup C)'$. Similarly, the green region, u is the elements that are in B but not in A or C , i.e. $B \cap (A \cup C)'$. Lastly, the blue region marked with y is the elements that are in C but not in A or B , the set $C \cap (A \cup B)'$.

The final region, z , outside of all the circles is the elements that are not in A nor in B nor in C . It is the set $(A \cup B \cup C)'$.

We will return to these ideas with a later example, but first we need a few more ideas.

Size of a set. The *cardinality* of a set is the number of elements contained in the set and is denoted $n(A)$.

Example 12. If $A = \{\text{egg, milk, flour, sugar, butter}\}$, then $n(A) = 5$. Note, the empty set, $\{\}$, has no elements, so $n(\{\}) = 0$.

Properties. *de Morgan's Laws*

$$(A \cup B)' = A' \cap B', \quad \text{and} \quad (A \cap B)' = A' \cup B'$$

I will illustrate the first law using Venn diagrams and leave the illustration of the other equation to the reader.

Consider the Venn diagram with two sets using the colors blue for A and orange for B , from example 9, we know that $A \cup B$ is the part containing any color.

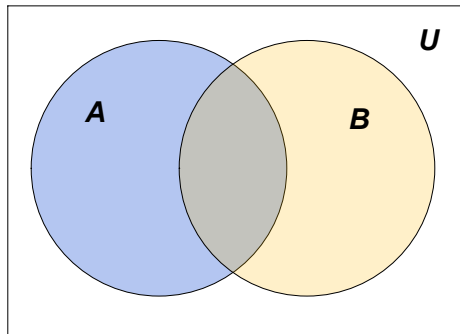


FIGURE 1. $A \cup B$

So $(A \cup B)'$ is the white part of the current diagram that is outside of both circles. The diagram to illustrate this is given below, the set is colored in red.

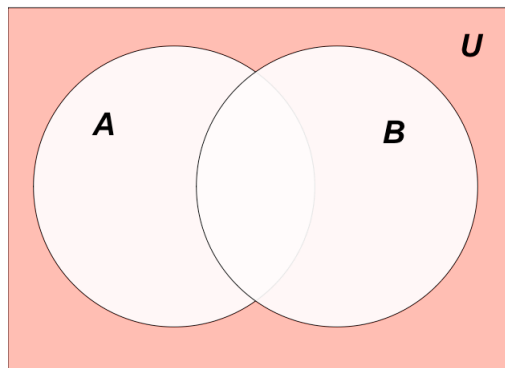
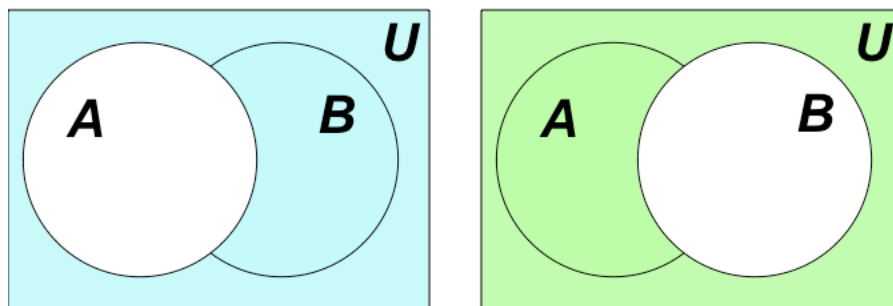


FIGURE 2. $(A \cup B)'$

Figure 1 and 2 demonstrate the left side of the equation $(A \cup B)' = A' \cap B'$. To demonstrate the right hand side of the equation, we will start with diagrams for A' and B' .

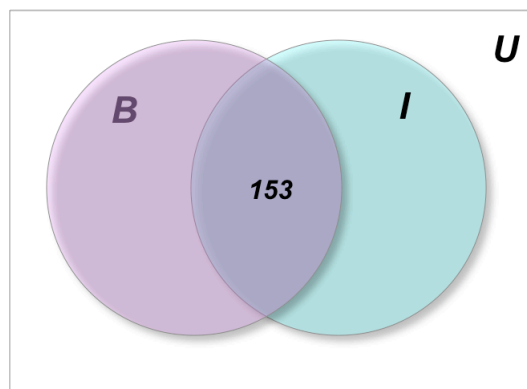
Here we see that everything outside of A is blue and everything outside of B is green. If we lay one diagram over the other, any part containing both colors will be the intersection of those two sets, $A' \cap B'$. The result will be Figure 2.

FIGURE 3. A' B'

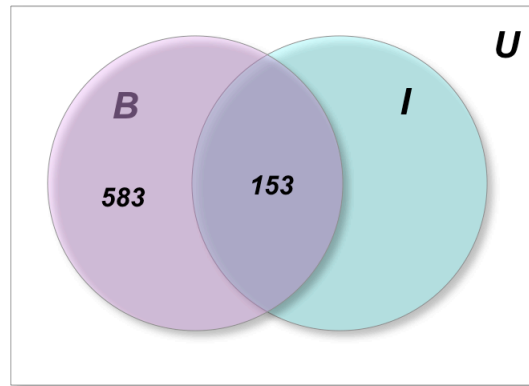
Applications. In this section, I will illustrate the use of Venn diagrams in some examples. Often we use the cardinality of a set for the Venn diagram rather than the actual objects.

Example 13. Two programs were broadcast on television at the same time; one was the Big Game and the other was Ice Stars. The Nelson Ratings Company uses boxes attached to television sets to determine what shows are actually being watched. In its survey of 1000 homes at the midpoint of the broadcasts, their equipment showed that 153 households were watching both shows, 736 were watching the Big Game and 55 households were not watching either. How many households were watching only Ice Stars? What percentage of the households were not watching either broadcast?

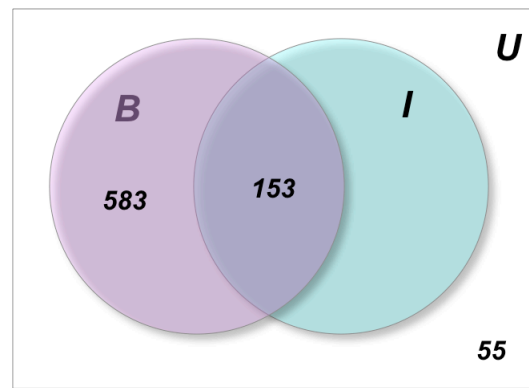
We begin by constructing a Venn diagram, we will use B for the Big Game and I for Ice Stars. Rather than entering the name of every household involved, we will put the cardinality of each set in its place within the diagram. So, since they told us that 153 households were watching both broadcasts, we know that $n(B \cap I) = 153$, this number goes in the dark purple "football" area.



We are told that 736 were watching the Big Game, $n(B) = 736$, since we already have 153 in that part of B that is in common with I , the remaining part of B will have $736 - 153 = 583$. This tells us that 583 households were watching only the Big Game.



We are also told that 55 households were watching neither program, $n((A \cup B)') = 55$, so that number goes outside of both circles.



Finally, we know that the total of everything should be 1000, $n(U) = 1000$. Since only one area does not yet contain a number it must be the missing amount to add up to 1000. We add the three numbers that we have, $583 + 153 + 55 = 791$, and subtract that total from 1000, $1000 - 791 = 209$, to get the number that were watching only Ice Stars. Filling in this number, we have a complete Venn diagram representing the survey.

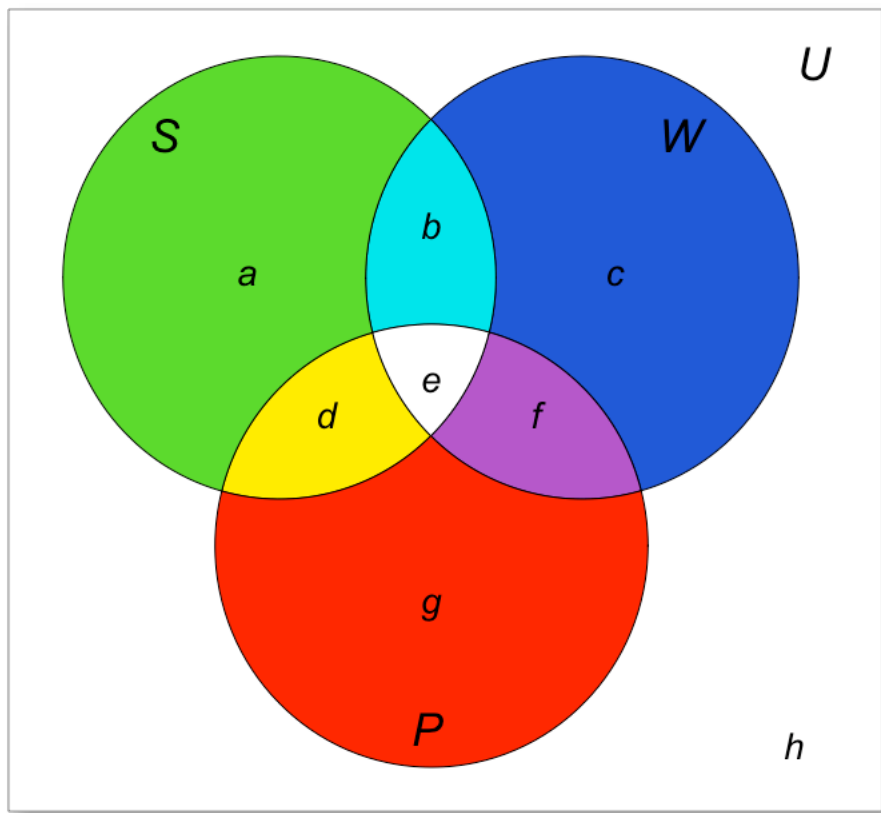


Now, we have the information needed to answer any questions about the survey results. In particular, we were asked how many households were watching only Ice Stars, we found this number to be 209. We were also asked what percentage of the households were watching only the Big Game. The number watching only the game was found to be 583, so we compute the percentage, $(583/1000) * 100\% = 58.3\%$.

Example 14. In a recent survey people were asked if they took a vacation in the summer, winter, or spring in the past year. The results were 73 took a vacation in the summer, 51 took a vacation in the winter, 27 took a vacation in the spring, and 2 had taken no vacation. Also, 10 had taken vacations at all three times, 33 had taken both a summer and a winter vacation, 18 had taken only a winter vacation, and 5 had taken both a summer and spring but not a winter vacation.

- (1) How many people had been surveyed?
- (2) How many people had taken vacations at exactly two times of the year?
- (3) How many people had taken vacations during at most one time of the year?
- (4) What percentage had taken vacations during both summer and winter but not spring?

To begin to answer these questions we will make a Venn diagram representing the information. Using S for summer, W for winter, and P for spring our diagram looks like the following.



We start by writing down all of the information given.

$$n(S) = 73$$

$$n(W) = 51$$

$$n(P) = 27$$

$$n((S \cup W \cup P)') = 2$$

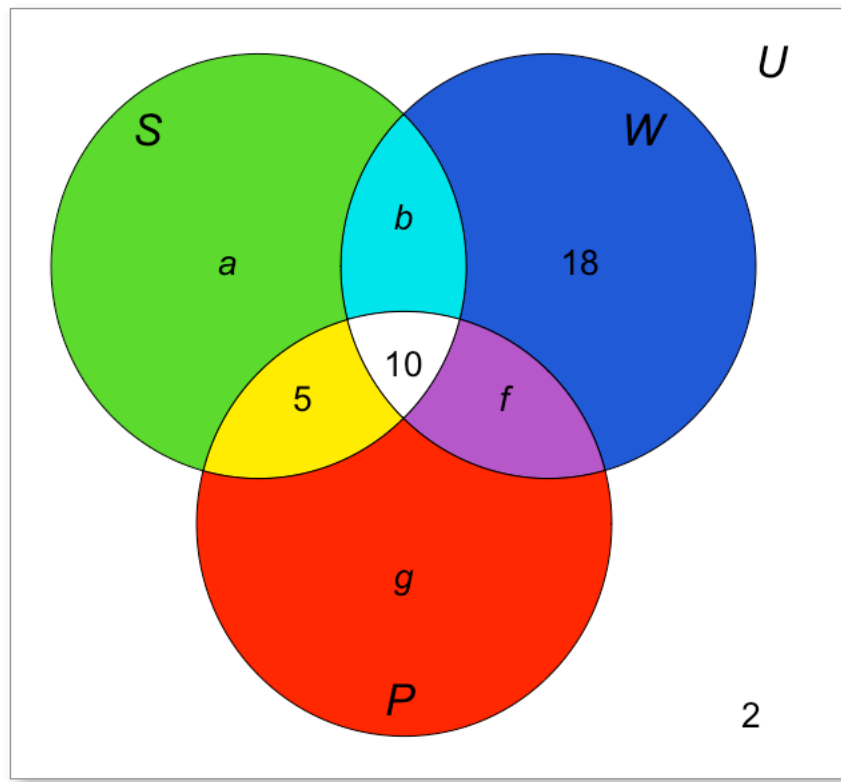
$$n(S \cap W \cap P) = 10$$

$$n(S \cap W) = 33$$

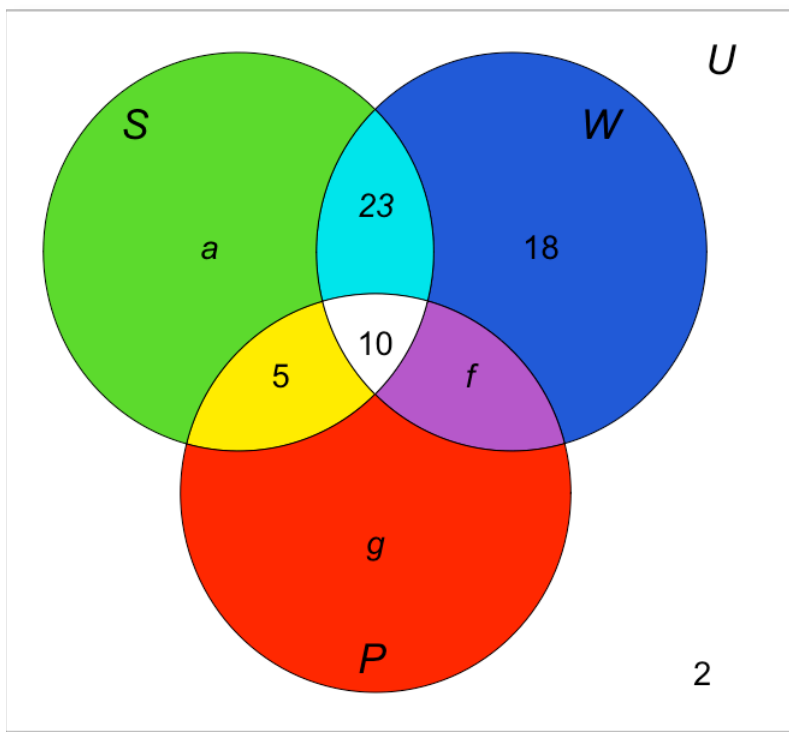
$$n(W \cap (S \cup P)') = 18$$

$$n(S \cap P \cap W') = 5$$

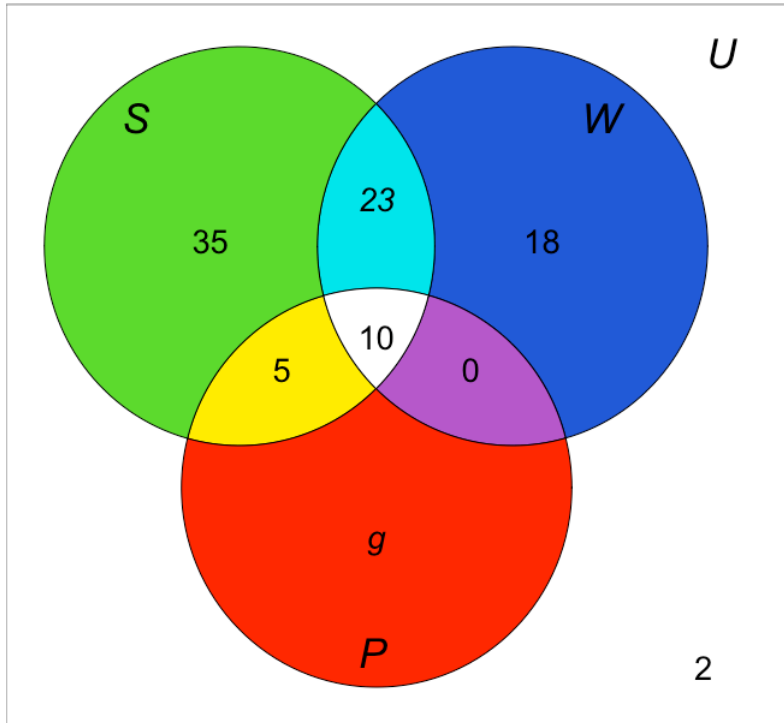
Look at this information to see if any of it can be entered into the diagram with no further work. It is best to start with the center if possible, and then the remainder of the "triangular" pieces. Our information given, tells us that $e = 10$, $h = 2$, $c = 18$, and $d = 5$.



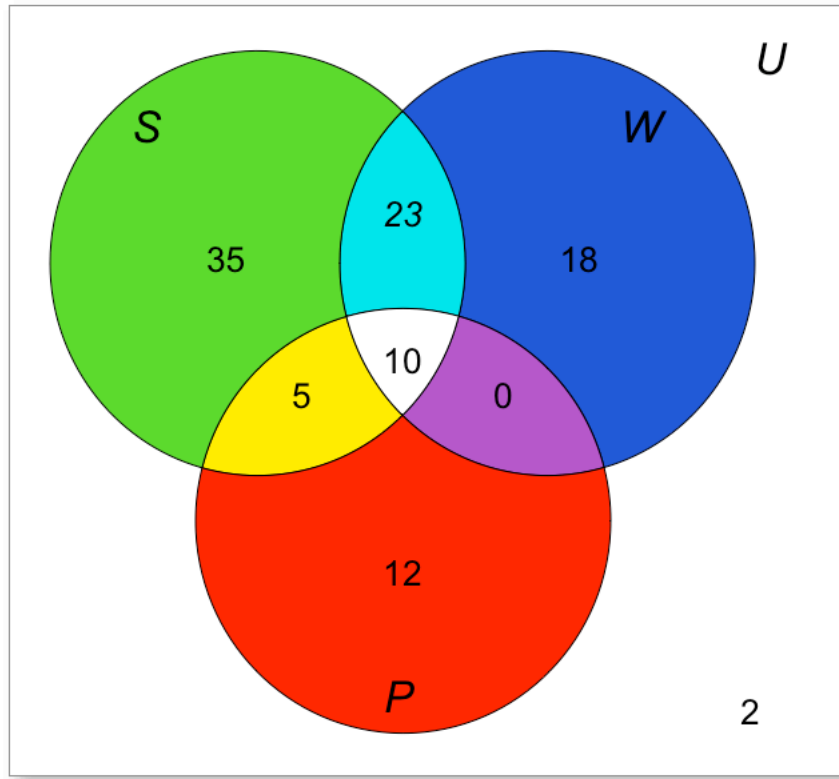
We know that the football shape that is cyan and white is supposed to have 33 people, since 10 are accounted for in the white portion, $33 - 10 = 23 = b$.



We now have three of the four pieces that make up S and we know the total of all of the pieces of S , so we find the green piece, a , $73 - (23 + 10 + 5) = 35 - a$. We also know three of the four pieces that make up W , this gives us the purple piece, f , $51 - (23 + 10 + 18) = 0 = f$.



Now we have three of the four pieces of P and only need to find the red piece g , $27 - (5 + 10) = 12 = g$, to complete the Venn diagram and begin answering the questions.



Now, to answer the questions

- (1) This is asking the size of our universal set, so we add *all* of the numbers in our diagram, $35 + 23 + 10 + 5 + 18 + 12 + 2 = 103$, thus 104 people were surveyed.
- (2) Those who have taken vacations exactly two times of the year would be the triangular pieces that are the intersection of only two of the sets, this is all of the triangular pieces except the white one in the center, $23 + 5 + 0 = 28$, hence 28 people took vacations at exactly two times of the year.
- (3) Those people who took vacations at most one time of the year either took a vacation during exactly one of the seasons, these are the green, blue, and red pieces, or had not taken a vacation at all, the number outside of all the circles. Adding these together, $35 + 18 + 12 + 2 = 67$, we get that 67 people had taken at most one vacation.
- (4) The number that took vacations during both summer and winter but not spring is the cyan section. Thus the percentage who took vacations only during those two seasons is $(23/104) * 100\% = 22.115\%$.

COUNTING

Fundamental Counting Principle. I will introduce the first counting technique with an example.

Example 15. Suppose a cafeteria offers a \$5 lunch special which includes one entree, a beverage, and a side. For the entree, you can choose to have either soup or a sandwich; the beverage choices are soda, lemonade, or milk; and the side choices are chips or cookies. How many different lunch combinations are there?

We will list the possibilities:

<i>soup, soda, chips</i>	<i>soup, soda, cookies</i>
<i>soup, lemonade, chips</i>	<i>soup, lemonade, cookies</i>
<i>soup, milk, chips</i>	<i>soup, milk, cookies</i>
<i>sandwich, soda, chips,</i>	<i>sandwich, soda, cookies</i>
<i>sandwich, lemonade, chips</i>	<i>sandwich, lemonade, cookies</i>
<i>sandwich, milk, chips</i>	<i>sandwich, milk, cookies</i>

Notice that this task involved a sequence of choices. We had to make a sequence of three choices to complete the task. For each choice we had options. If we think of the task as a sequence of boxes to fill, we can set it up as:



The ***Fundamental Counting Principle*** gives us a quicker way to count up the number of ways to complete the task. If a task requires a sequence of choices, then the number of ways to complete the task is to multiply together the number of options for each choice.

Example 16. In our previous example, we counted up the number of ways to make a lunch combination by listing them all out. With the Fundamental Counting Principle, we could have simply multiplied:

$$\boxed{2} \times \boxed{3} \times \boxed{2} = 12.$$

entree beverage side

Example 17. How many license plates can be made if each is to be three digits followed by 3 letters. The plate number cannot begin with a 0.

We can think of this as a sequence of tasks and apply the Fundamental Counting Principle:

$$\boxed{9} \times \boxed{10} \times \boxed{10} \times \boxed{26} \times \boxed{26} \times \boxed{26} = 15,818,400.$$

digit digit digit digit digit digit

Factorial. Before we can continue on to our next counting technique, we will need to learn a new idea and notation. The idea is called the ***factorial***, it has the notation ***!***. It is easiest to understand the idea by looking at the pattern.

$$\begin{aligned}
 0! &= 1 \\
 1! &= 1 \\
 2! &= 2 \cdot 1 \\
 3! &= 3 \cdot 2 \cdot 1 \\
 4! &= 4 \cdot 3 \cdot 2 \cdot 1
 \end{aligned}$$

The idea can be expressed in general as:

$$n! = n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1.$$

We could stop the expansion process at any point and indicate the remainder of the factorial in terms of a lower factorial.

$$\begin{aligned}
 5! &= 5 \cdot 4! \\
 &= 5 \cdot 4 \cdot 3! \\
 &= 5 \cdot 4 \cdot 3 \cdot 2! \\
 &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1! \\
 &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1
 \end{aligned}$$

The ability to expand any factorial partially can be an aid in simplifying expressions involving more than one factorial.

Example 18. Compute, by expanding and simplifying,

$$\frac{12!}{9!}.$$

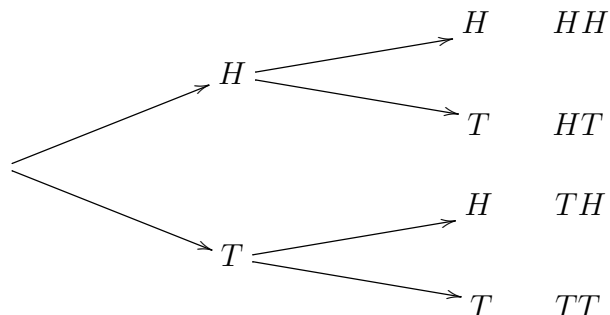
Solution:

$$\begin{aligned}
 \frac{12!}{9!} &= \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!} && \text{(by expanding)} \\
 &= \frac{12 \cdot 11 \cdot 10 \cdot \cancel{9!}}{\cancel{9!}} && \text{(cancelling like terms)} \\
 &= 12 \cdot 11 \cdot 10 = 1320
 \end{aligned}$$

Tree Diagram. Another tool that is very useful in counting is called a *tree diagram*. This is a picture that branches for each option in choice.

Example 19. If I toss a penny and a nickel, how many possible outcomes are there?

Using the symbols H for heads and T for tails, we get the following tree:



The first coin has a possible outcome of H or T , this is represented with the first branch. For each branch here, the second coin has the outcome of H or T , represented by the second set of branches. At the end of each branch, I have listed the result of starting at the beginning and following along the branches to the end. There are 4 ends, so the total number of possible outcomes of tossing a penny and a nickel is 4.

Permutations and Combinations. Next we will consider the difference in the following tasks and learn how to count them.

The first task is to take 3 books from a pile of 8 distinct books and line them up. The second task is to select 3 of the 8 distinct books.

The first thing to notice is that we can distinguish each object from any other and that we cannot replace any book that has been used, this is called *without replacement*. Both of our tasks have this feature. Next, we note that the first task is line up the books, thus order makes a difference; {math, art, english} would look different from {art, math, english}. So for task one, ORDER MATTERS. In the second task, we simply choose a group of 3 with no arranging, so for task two, ORDER DOES NOT MATTER.

This leads us to the definitions for these situations. A *permutation* is an arrangement of objects. A *combination* is a collection of objects.

Next, we will learn how to count the number of permutations and combinations.

The number of permutations (arrangements) without replacement of r objects from a group of n distinct objects is denoted $P(n, r)$ or ${}_n P_r$ and is calculated with the formula:

$$P(n, r) = \frac{n!}{(n - r)!}.$$

The number of combinations (groups) without replacement of r objects chosen from a group of n distinct objects is denoted $C(n, r)$ or ${}_n C_r$ and is calculated with the formula

$$C(n, r) = \frac{n!}{r!(n - r)!}.$$

Example 20. Consider the set $\{a, b, 5\}$.

- (1) How many permutations of 2 of the objects are possible?

Solution: The first solution is to simply list out the permutations:

$$ab, a5, ba, b5, 5a, 5b$$

to see that there are a total of 6. The second solution is to use the formula where r is 2 (the number of objects being arranged) and n is 3 (the number of objects from which we are selecting those to arrange). This gives the result:

$$P(3, 2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{6}{1} = 6.$$

- (2) How many groups of two of the objects are there?

Solution: Again, our first solution will be to list the possibilities:

$$\{a, b\}, \{a, 5\}, \{b, 5\}$$

to see that there are only three ways to choose two of the objects. The second solution is to use the formula with $r = 2$ (the number of objects to be chosen) and $n = 3$ (the number of objects from which we are choosing). This gives the result:

$$C(3, 2) = \frac{3!}{2!(3-2)!} = \frac{3!}{2!1!} = \frac{6}{2 \cdot 1} = \frac{6}{2} = 3.$$

Example 21. A freshman class consists of 40 students, 30 of which are women. The class needs to select a committee of 7 to represent them in the student senate. How many committees are possible if

- (1) the committee must have exactly 5 women?

Solution: If the committee has exactly 5 women then it is composed of 5 women and 2 men, meaning that we must do two tasks in order to do the entire selection. So we need to choose 5 women and 2 men.

$$(\text{number of ways to choose 5 women}) \times (\text{number of ways to choose 2 men})$$

Now, we have to choose 5 women from the 30 available and order does not matter; this is ${}_{30}C_5$. We also need to choose 2 men from the 10 men available where order does not matter; this is ${}_{10}C_2$. The number of ways to choose a committee with exactly 5 women is

$${}_{30}C_5 \times {}_{10}C_2 = 142506(45) = 6,412,770.$$

- (2) the committee must have at least 5 women?

Solution: In this case, we have more than one way to satisfy the condition, the choices are: (5 women and 2 men) or (6 women and 1 man) or (7 women). The first is the number that we computed in the previous part. Using the same idea for the other two options, we have

$${}_{30}C_5 \times {}_{10}C_2 + {}_{30}C_6 \times {}_{10}C_1 + {}_{30}C_7 = 14,386,320.$$