Probability

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Vocabulary

In order to discuss probability we will need a fair bit of vocabulary.

Probability is a measurement of how likely it is for something to happen.

An ***experiment*** is a process that yields an observation.

Example 1. Experiment 1. The experiment is to toss a coin and observe whether it lands heads or tails.

Experiment 2. The experiment is to toss a die and observe the number of spots.

An ***outcome*** is the result of an experiment.

Example 2. Experiment 1. One possible outcome is heads, which we will designate H.

Experiment 2. One possible outcome is 5.

Sample space is the set of all possible outcomes, we will usually represent this set with S.

Example 3. Experiment 1. $S = \{H, T\}$

Experiment 2. $S = \{1, 2, 3, 4, 5, 6\}$.

An ***event*** is any subset of the sample space. Events are frequently designated as E or E_1, E_2, etc. if there are more than one.

Example 4. Experiment 1. One possible event is landing heads up. $E = \{H\}$.

Experiment 2. One event might be getting an even number, a second event might be getting a number greater than or equal to 5. $E_1 = \{2, 4, 6\}, \quad E_2 = \{5, 6\}$.

A ***certain event*** is guaranteed to happen, a "sure thing". An ***impossible event*** is one that cannot happen.

Example 5. From our Experiment 1. If the event E_1 is landing heads up or landing tails up, then $E_1 = \{H, T\}$ and we can see that E_1 is a certain event. If the event E_2 is getting a 6, then $E_2 = \{\}$ and we can see than E_2 is impossible.

Outcomes in a sample space are said to be ***equally likely*** if every outcome in the sample space has the same chance of happening, *i.e.* one outcome is no more likely than any other.

BASIC COMPUTATION OF A PROBABILITY

If E is an event in an equally likely sample space, S, then the probability of E, denoted $p(E)$ is computed

$$p(E) = \frac{n(E)}{n(S)}$$

Example 6. Experiment 1. $E_1 = \{H, T\}$, so $p(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{2} = 1$.

Experiment 2. $E_2 = \{5, 6\}$, so $p(E_2) = \frac{n(E_2)}{n(S)} = \frac{2}{6} = \frac{1}{3}$.

You should note that since $n(\emptyset) = 0$, $p(\emptyset) = 0$ and since $\frac{n(S)}{n(S)} = 1$, $p(S) = 1$.

Relative frequency is the number of times a particular outcome occurs dvided by the number of times the experiment is performed.

Example 7. Suppose you toss a fair coin 10 times and observe the results: heads occurred 6 times and tails occurred 4 times. In this case the relative frequency of heads would be $\frac{6}{10} = 0.6$.

Now suppose you toss a fair coin 1000 times and observe the results: heads occurred 528 times and tails occured 472 times. Now the relative frequency of heads is $\frac{528}{1000} = 0.528$.

Something which can be observed from the previous two experiments is called the Law of Large Numbers. *The Law of Large Numbers*: If an experiment is repeated a large number of times, the relative frequency tends to get closer to the probability.

Example 8. In the previous example, we observed the outcome heads. This was a fair coin so we know $p(H) = 0.5$. Notice that when we performed the experiment 1000 times, the relative frequency was closer to 0.5 than when we performed the experiment 10 times.

ODDS

The *odds for* an event, E, with equally likely outcomes are:

$$o(E) = n(E) : n(\overline{E})$$

and the *odds against* the event E are:

$$o(\overline{E}) = n(\overline{E}) : n(E)$$

. Note, these are read as a to b.

Example 9. Toss a fair coing twice and observe the outcome. The sample space is

$$S = \{HH, HT, TH, TT\}.$$

If E is getting two tails, then

$$o(E) = 1 : 3,$$

the odds for E are 1 to 3. Also

$$o(\overline{E}) = 3 : 1,$$

the odds against E are 3 to 1.

Some Applications

Probability is used in genetics. When Mendel experimented with pea plants he discovered that some genes were dominant and others were recessive. This means that given one gene from each parent, the dominant trait will show up unless a recessive gene is received from each parent. This is often demonstrated by using a "Punnett square". Here is a typical Punnett square:

	R	**R**
w	wR	wR
w	wR	wR

The letters along the top of the table represent the gene contribution from one parent and the letters down the left-hand side of the table represent the gene contribution from the other parent. Each cell of the table contains a genetic combination for a possible offspring. This particular Punnett square represent the crossing of a pure red flower pea with a pure white flower pea. The offspring will each inherit one of each gene; since red is dominant here, all offspring will be red.

Example 10. Suppose we cross a pure red flower pea plant with one of the offspring that has one of each gene. The resulting Punnett square would be

	R	**R**
w	wR	wR
R	RR	RR

From this we can see that the probability of producing a pure red flower pea is $\frac{2}{4} = \frac{1}{2}$, and we can see that each offspring would produce red flowers.

Example 11. This time we will cross two of the offspring which have one of each gene. The Punnett square is

	w	**R**
w	ww	wR
R	Rw	RR

Here we can find that the probability of an offspring producing white flowers is $\frac{1}{4}$.

Example 12. Sickle cell anemia is inherited. This is a co-dominant disease. A person with two sickle cell genes will have the disease while a person with one sickle cell gene will have a mild anemia called sickle cell trait. Suppose a healthy parent (no sickle cell gene) and a parent with sickle cell trait have children. Use a Punnett square to determine the following probabilities.

    (1) the child has sickle cell
    (2) the child has sickle cell trait
    (3) the child is healthy

Solution: We will use S for the sickle cell gene and N for no sickle cell gene. Our Punnett square is

	N	**N**
S	SN	SN
N	NN	NN

(1) No offspring have two sickle cell genes so this probability is 0.
(2) We see that two of the offspring will have one of each gene, so this probability is $\frac{2}{4} = \frac{1}{2}$.
(3) Two of the children will have no sickle cell genes, this probability is $\frac{2}{4} = \frac{1}{2}$.

BASIC PROPERTIES OF PROBABILITY

We will need to start with one more new term. Two events are ***mutually exclusive*** if they cannot both happen at the same time, *i.e.* $E \cap F = \emptyset$. So, if $p(A \cap B) = 0$ then A and B are mutually exclusive,

We have the following rules for probability:

> **Basic Probability Rules**
> $p(\emptyset) = 0$,
> $p(S) = 1$,
> $0 \le p(E) \le 1$

Example 13. Roll a die and observe the number of dots; $S = \{1, 2, 3, 4, 5, 6\}$. Given E is the event that you roll a 15, F is the event that you roll a number between 1 and 6 inclusive, and G is the event that you roll a 3. Find $p(E)$, $p(F)$, and $p(G)$.

Solution: Since 15 is not in the sample space $E = \emptyset$ and $p(E) = 0$. $F = S$, so $p(F) = 1$. Lastly, $G = \{3\}$, so $p(G) = \frac{n(G)}{n(S)} \frac{1}{6}$.

Example 14. Roll a pair of dice, one red and one white. Find the probabilities:

(1) the sum of the pair is 7
(2) the sum is greater than 9
(3) the sum is not greater than 9
(4) the sum is greater than 9 and even
(5) the sum is greater than 9 or even
(6) the difference is 3

Solution: This sample space is too large to list as there are $6 \times 6 = 36$ outcomes. Hence, we will just count up the number of ways to achieve each event and them divide by 36.

(1) The pairs that sum to 7 are $\{(6, 1), (1, 6), (5, 2), (2, 5), (4, 3), (3, 4)\}$, hence this probability is $\frac{6}{36} = \frac{1}{6}$.
(2) Sums greater than 9 are 10, 11 and 12, the pairs meeting this condition are $\{(6, 4), (4, 6), (5, 5), (6, 5), (5, 6), (6, 6)\}$, this gives the probability as $\frac{6}{36} = \frac{1}{6}$.
(3) The sum is not greater than 9 would be all of those not included in the previous part, so there must be $36 - 6 = 30$ outcomes in this set. Hence, this probability is $\frac{30}{36} = \frac{5}{6}$.
(4) Those with a sum greater than 9 and even would be those that sum to 10 or 12, $\{(6, 4), (4, 6), (5, 5), (6, 6)\}$. Since this set contains 4 outcomes, this probability is $\frac{4}{36} = \frac{1}{9}$.

(5) Half of the pairs would sum to an even number and part (3) gives us that there are 6 with sums greater than 9. Note that (4) gives us the number in the intersection. Using the counting formula $n(A \cup B) = n(a) + n(B) - n(A \cap B)$, this set contains $18 + 6 - 4 = 20$ and the probability is $\frac{20}{36} = \frac{5}{9}$.

(6) Those pairs whose difference is 3 are $\{(6,3), (3,6), (5,2), (2,5), (4,1), (1,4)\}$, hence the probability is $\frac{6}{36} = \frac{1}{6}$.

We have two more formulae for probability that will be useful.

> **Basic Probability Rules**
> $p(E) + p(\overline{E}) = 1,$
> $p(E \cup F) = p(E) + p(F) - p(E \cap F)$

Use of Venn Diagrams for Probability. It is often helpful to put the information given in a Venn Diagram to organize the information and answer questions. The following is an example of such a case.

Example 15. Zaptronics makes CDs and their cases for several music labels. A recent sampling indicated that there is a 0.05 probability that a case is defective, a 0.97 probability that a CD is not defective, and a 0.07 probability that at lease one of them is defective.

  (1) What is the probability that both are defective?
  (2) What is the probability that neither is defective?

Solution: We start by writing down the information given. Let C represent a defective case and let D represent a defective CD. Then we are given:

$$p(C) = 0.05, \qquad p(\overline{D}) = 0.97, \qquad p(C \cup D) = 0.07$$

Using the property $p(E) + p(\overline{E}) = 1$, we get $p(D) = 1 - p(\overline{D}) = 1 - 0.97 = 0.03$. Using the property $p(E \cup F) = p(E) + p(F) - p(E \cap F)$, we get $p(C \cap D) = p(C) + p(D) - p(C \cup D) = 0.05 + 0.03 - 0.07 = 0.01$. Put this together in a Venn diagram and recall that $p(U) = 1$.

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From this we can see the answers we need.

  (1) This is the middle football shape so, 0.01
  (2) This is everything outside of the circles, so 0.93

EXPECTED VALUE

Once again, we will need some new vocabulary. The ***payoff*** is the value of an outcome.

Example 16. You toss a coin, if it comes up heads you win \$1, if it comes up tails, you pay me \$0.50. The outcome H, has the payoff 1. The outcome T, has the payoff -0.5.

The ***expected value*** of an event is a long term average of payoffs of an experiment. If the experiment were repeated often enough, the actual profit/loss will get close to the expected value. To compute the expected value, you first multiply each payoff by the probability of getting that payoff. Once you have done all of the multiplications, you add your results together. This sum is the expected value of an experiment.

Let $m_1, m_2, m_3, \ldots, m_k$ be the payoffs associated with the k outcomes of an experiment. Let $p_1, p_2, p_3, \ldots, p_k$, repectively, be the probabilities of those outcomes. Then the expected value is found as follows.

> **Expected Value:** $\quad E = m_1 p_1 + m_2 p_2 + m_3 p_3 + \cdots + m_k p_k$

In mathematics, the capital greek letter sigma, Σ, is used to tell us to add up the things that follow. Using this idea, an informal but perhaps more palatable form of the formula for expected value is

$$E = \Sigma(\text{probability}) \times (\text{payoff}).$$

Some example should be done here.

CONDITIONAL PROBABILITY

We will introduce the idea of conditional probability with an example.

Example 17. Two coins are tossed. Event E is getting exactly one tail. Event F is getting at least one tail. The sample space for the experiment is

$$S = \{HH, HT, TH, TT\}.$$

The event

$$E - \{HT, TH\}$$

has probability $p(E) = 2/4$.

If I tell you that at least one of the coins is showing a tail, then you know that event F has occurred and that the outcome HH is no longer possible. Hence, we effectively reduce the size of the sample space to

$$F = \{HT, TH, TT\}.$$

With F as the sample space, the probability of event $E = 2/3$. Thus, the probability of one event occurring is changed by knowing that another event has already occured. This is conditional probability.

Conditional probability is a probability that is based on knowing that some event within a sample space has already occurred. The notation for the probability of the event E occurring when it is known that the event F has occurred is $p(E|F)$ and is read "the probability of E given F". The formula for computing the conditional probability is

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

Example 18. In our previous example we found $p(E|F) = 2/3$ by reducing the size of our sample space and using the basic probability formula. We could have found it using the formula.

$$E \cap F = \{HT, TH\} \quad \text{so} \quad p(E \cap F) = 1/2$$

Using the formula, we get

$$p(E|F) = \frac{1/2}{3/4} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3},$$

which is exactly the same value we got by counting.

Some examples should be inserted here.

Associated with conditional probability is the idea of independence of two events. Two events are ***independent events*** if knowing that one of them has occurred does not change the probability that the other occurs; $p(E|F) = p(E)$.

Some examples need to be inserted here.

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