

Dimensional Analysis and Exponential Models

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1. DIMENSIONAL ANALYSIS

The purpose of this section is to convert between various types of units; feet to miles, kilometers per hour to meters per second, square feet to square inches, cubic meters to cubic centimeters; etc. *Dimensional analysis* is the process of using a standard conversion to create a fraction, including units in that fraction and canceling units in the same way that variables are cancelled.

We do need to know some relationships between units:

$$\begin{array}{ll} 1 \text{ mile} = 5280 \text{ feet}, & 3 \text{ feet} = 1 \text{ yard}, \\ 1000 \text{ meters} = 1 \text{ kilometer}, & 100 \text{ centimeters} = 1 \text{ meter} \end{array}$$

If you write out the full process and make sure that the units cancel leaving only what you want, you should be successful in converting units. The remainder of this section is examples.

Example 1. Convert 6 feet to yards.

Solution:

$$6 \text{ feet} = \frac{6 \cancel{\text{feet}}}{1} \cdot \frac{1 \text{ yard}}{3 \cancel{\text{feet}}} = \frac{6 \text{ yard}}{3} = 2 \text{ yards}$$

Example 2. Convert 765 inches per second to miles per hour:

Solution:

$$\begin{aligned} 765 \text{ in/sec} &= \frac{765 \cancel{\text{inches}}}{\text{second}} \cdot \frac{\cancel{\text{foot}}}{12 \cancel{\text{inches}}} \cdot \frac{1 \text{ mile}}{5280 \cancel{\text{feet}}} = \frac{765 \text{ miles}}{12 \cdot 5280 \text{ second}} \\ &= \frac{17 \text{ miles}}{1408 \cancel{\text{second}}} \cdot \frac{60 \cancel{\text{second}}}{1 \cancel{\text{minute}}} \cdot \frac{60 \cancel{\text{minute}}}{1 \text{ hour}} = \frac{17 \cdot 60 \cdot 60 \text{ miles}}{1408 \text{ hour}} \\ &= \frac{7650}{176} \text{ mi/hr} = \frac{3925}{88} \text{ mi/hr} \approx 43.4659 \text{ mi/hr} \end{aligned}$$

Example 3. Convert 40 square feet to square inches.

Solution:

$$40 \text{ ft}^2 = \frac{40 \cancel{\text{ft}} \cdot \cancel{\text{ft}}}{1} \cdot \frac{12 \text{ in}}{1 \cancel{\text{ft}}} \cdot \frac{12 \text{ in}}{1 \cancel{\text{ft}}} = 5760 \text{ in}^2$$

Example 4. Convert 2 cubic yards to cubic feet.

Solution:

$$2 \text{ yd}^3 = \frac{2 \cancel{\text{yd}} \cdot \cancel{\text{yd}} \cdot \cancel{\text{yd}}}{1} \cdot \frac{3 \text{ ft}}{\cancel{\text{yd}}} \cdot \frac{3 \text{ ft}}{\cancel{\text{yd}}} \cdot \frac{3 \text{ ft}}{\cancel{\text{yd}}} = 2 \cdot 3 \cdot 3 \cdot 3 \text{ ft}^3 = 54 \text{ ft}^3$$

Example 5. Convert 75 centimeters per second to meters per hour:

Solution:

$$\begin{aligned} 75 \text{ cm/sec} &= \frac{75 \cancel{\text{ cm}}}{\text{second}} \cdot \frac{1 \text{ meter}}{100 \cancel{\text{ cm}}} \cdot \frac{60 \cancel{\text{ second}}}{1 \cancel{\text{ minute}}} \cdot \frac{60 \cancel{\text{ minute}}}{1 \text{ hour}} \\ &= \frac{75 \cdot 60 \cdot 60 \text{ meters}}{100 \text{ hour}} = 2736 \text{ m/hr} \end{aligned}$$

For more practice and to ensure that the process is clear, we will also do some with unfamiliar (made up) units.

Example 6. Suppose we are given that 13 horks is equivalent to one plop, 7 plops are equivalent to one wooze, 5 hons is equal to one slop and 11 slops are equal to one murk. Convert:

- (1) 80 horks to woozes
- (2) 9 square plops to square horks
- (3) 2 cubic woozes to cubic plops
- (4) 12 horks per hon to plops per murk
- (5) 18 cubic horks per murk to cubic plops per slop

Solution:

(1)

$$80 \text{ horks} = \frac{80 \cancel{\text{ hork}}}{1} \cdot \frac{\cancel{\text{ plop}}}{13 \cancel{\text{ hork}}} \cdot \frac{\text{wooze}}{7 \cancel{\text{ plop}}} = \frac{80}{13 \cdot 7} \text{ woozes} = \frac{80}{91} \text{ woozes} \approx 0.8791 \text{ woozes}$$

(2)

$$9 \text{ plops}^2 = \frac{9 \cancel{\text{ plop}}}{1} \cdot \frac{13 \cancel{\text{ hork}}}{\cancel{\text{ plop}}} \cdot \frac{13 \cancel{\text{ hork}}}{\cancel{\text{ plop}}} = 1521 \text{ horks}^2$$

(3)

$$2 \text{ woozes}^3 = \frac{2 \cancel{\text{ woozes}}}{1} \cdot \frac{7 \cancel{\text{ plop}}}{\cancel{\text{ wooze}}} \cdot \frac{7 \cancel{\text{ plop}}}{\cancel{\text{ wooze}}} \cdot \frac{7 \cancel{\text{ plop}}}{\cancel{\text{ wooze}}} = 686 \text{ plops}^3$$

(4)

$$12 \text{ horks/hon} = \frac{12 \cancel{\text{ horks}}}{\cancel{\text{ hon}}} \cdot \frac{\cancel{\text{ plop}}}{13 \cancel{\text{ hork}}} \cdot \frac{5 \cancel{\text{ hon}}}{\cancel{\text{ slop}}} \cdot \frac{11 \cancel{\text{ slop}}}{\text{murk}} = \frac{660}{13} \text{ plops/murk} \approx 50.7692 \text{ plops/murk}$$

(5)

$$18 \text{ horks}^3/\text{murk} = \frac{18 \cancel{\text{ horks}}^3}{\cancel{\text{ hon}}} \cdot \frac{\cancel{\text{ plop}}}{13 \cancel{\text{ hork}}} \cdot \frac{\cancel{\text{ plop}}}{13 \cancel{\text{ hork}}} \cdot \frac{\cancel{\text{ plop}}}{13 \cancel{\text{ hork}}} \cdot \frac{5 \cancel{\text{ hon}}}{\cancel{\text{ slop}}} \cdot \frac{11 \cancel{\text{ slop}}}{\text{murk}} = \frac{660}{13} \text{ plops/murk} \approx 50.7692 \text{ plops/murk}$$

2. EXPONENTIAL FUNCTIONS

2.1. Function Review. For our purposes, we will only consider functions which can be given as a formula and whose input is some subset of the real numbers. In this case, recall that a **function** is a rule that assigns a value (output) according to the formula for each number you put in (input). The function has some name, the input variable is usually x but need not be. Some examples of functions are:

$$\text{sam}(x) = 2x + 1, \quad \text{tax}(s) = 3s^3 + 2s, \quad g(t) = \sqrt{t}.$$

For the function *sam*, the input variable is x , for *tax* it is s , and for g it is t . This variable is also called the *independent variable*. Think of it as the one that you get to choose.

Once you have followed the formula to produce a value, this value is called the *function value* or the *dependent variable*. The process of following the formula for a particular input is called evaluating the function. Let us look at our function *sam*. Suppose we want to evaluate *sam* at 3, then we would write $sam(3)$. Once you have followed the formula, $2x + 1$ where x is 2, you get 5. Then we would write $sam(3) = 5$. Remember that the independent variable in the formula is a place holder for the actual value that you will put in.

Example 7. Evaluate:

- (1) $tax(-1)$,
- (2) $tax\left(\frac{1}{2}\right)$,
- (3) $tax(3)$,
- (4) $tax(\heartsuit)$

Solution:

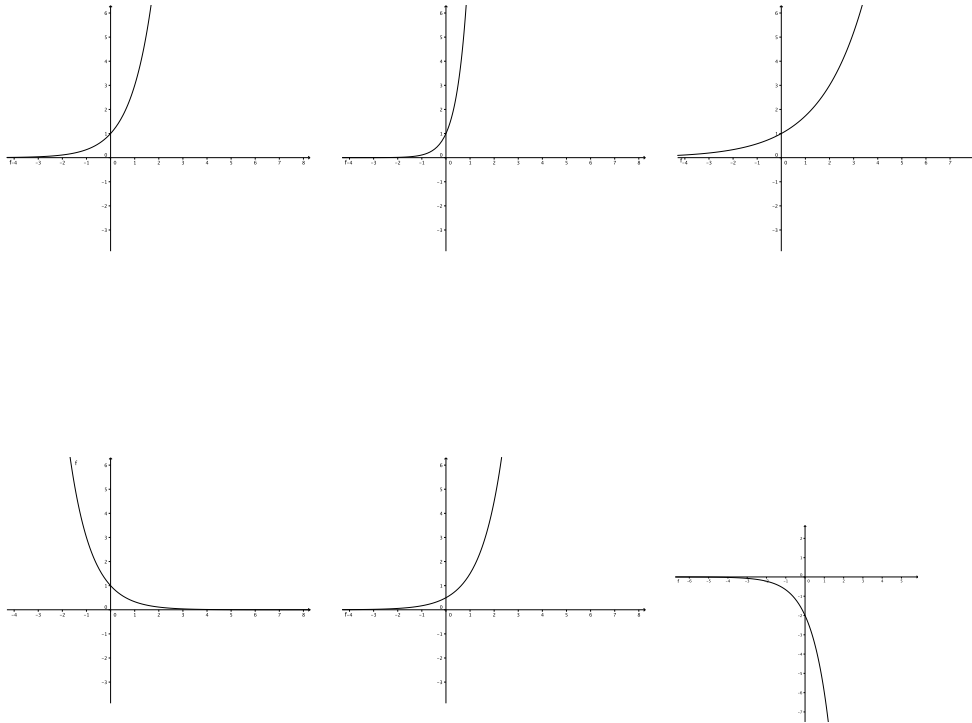
- (1) $tax(-1) = 3(-1)^2 - 2(-1) = 3(1) + 2 = 3 + 2 = 5$
- (2) $tax\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) = 3\left(\frac{1}{4}\right) - 1 = \frac{3}{4} - 1 = -\frac{1}{4}$
- (3) $tax(3) = 3(3)^2 - 2(3) = 3(9) - 6 = 27 - 6 = 21$
- (4) $tax(\heartsuit) = 3\heartsuit^2 - 2\heartsuit$

2.2. The exponential function. The functions that we will be concerned with will have the property that they grow or decay at a constant rate. Consider the following data collected in a laboratory:

<u>time</u>	<u>number of bacteria</u>
0	20
1 hr	35
2 hr	53
3 hr	79
4 hr	118

A little calculation shows us that this bacteria grows at a rate of 50% per hour. This type of function is an *exponential function* and they all have the formula $f(x) = ab^{kx}$ where $b > 0$. b is the *base* of the exponential and k is a constant determined by the growth rate. You often see this written as $y = ab^{kx}$.

There are several possible graphs for these functions depending on the values of a and k . Some examples of the graphs are given below. (Reminder, when you graph a function, the input is on the horizontal and the output on the vertical.)



Notice that four of these keep going up as you move from left to right. So the function values, y , get bigger as the input, x , gets bigger. Such functions are said to be **strictly increasing**. The other two functions have the property that y gets smaller as x gets bigger, these functions are said to be **strictly decreasing**. Recall, the y -intercept is the point where the graph crosses the x -axis. It is the point whose y value is the function evaluated at $x = 0$; hence the coordinates are $(0, f(0))$ (if the function is called f).

Let us examine the relationships between a and k that determine some of the traits of the graph. We will be grouping them according to attributes a and k .

The first set that we will look at all have $a > 0$.

$$f(x) = 3^x, \quad g(x) = 2 * 3^x, \quad h(x) = \frac{1}{2} * 3^x, \quad m(x) = 3^{-x}$$

We will look at the first three together on one graph so that we can compare the graphs and determine how the size of a effects the graph; see Figure 1. In Figure 1, f is graphed in green, g is graphed in red, and h is graphed in blue. In Figure 2, we have graphed f and m , with m being red, so that we can look at the effects of a negative k .

In Figure 1 see that all three of these functions will grow at the same rate, the difference is the y -intercept. The **initial value** is the value of the function when $x = 0$ or $f(0)$. So, the a in the function ab^{kx} is the initial value. In Figure 2, we see that the difference is whether the function is growing (increasing) or decaying (decreasing). Hence we conclude that the

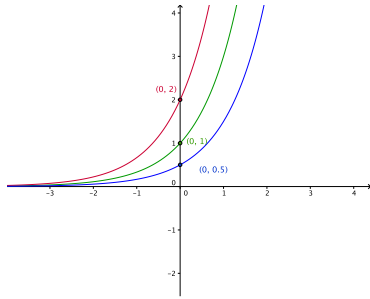


FIGURE 1. f , g , and h

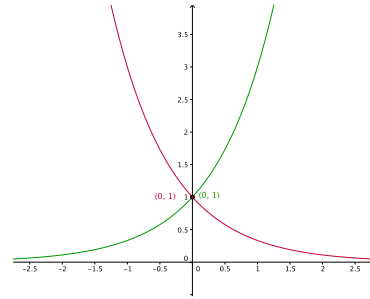


FIGURE 2. f and k

function ab^{kx} grows if $k > 0$ and decays if $k < 0$. We will need two more functions to get more information about what k does to the graph.

For this last comparison, we will look at the three functions

$$f(x) = 3^x, \quad p(x) = 3^{(1/2)x}, \quad q(x) = 3^{2x}$$

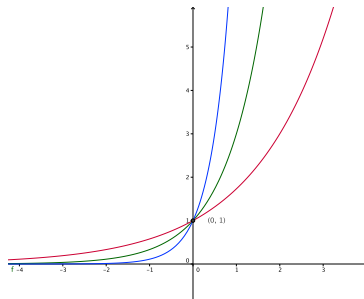


FIGURE 3. f , p , and q

In Figure 3, f is graphed in green, p with red, and q with blue. Here we see that the size of k effects how steep the graph is (how fast the function grows). The bigger k is, the faster the function grows.

The number e , often called *Euler's constant* is an important number in mathematics (similar to π) that cannot be written as a fraction or a root of any number. The numerical value

of e has been calculated to many decimal places, for our purposes (if we need a decimal approximation) at most 5 decimal places will be needed. We will approximate

$$e \approx 2.71828.$$

We have introduced e here because it turns out that every exponential $y = ab^{kx}$ can be written with the exponential function base e . Hence, from this point, we will only consider functions of the form $f(x) = ae^{kx}$.

2.3. Inverse Function. The exponential function has a property that allows for a related function, the inverse function. Recall that an *inverse* of a function undoes whatever the function does. For example, if the function multiplies what you put in by 2 and then subtracts 1, the inverse will add 1 and then divide by 2. For the exponential, the function that undoes raising e to a power is called the (natural) *logarithm* function, denoted $\ln(x)$. What we need to know about this function for our current purposes is actually quite limited. First, the graph of the function. In Figure 4, we have graphed $f(x) = 3 * e^{0.05x}$ and the inverse of f which we will call g .

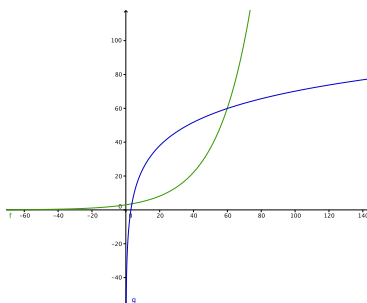


FIGURE 4. f and g

Both functions grow, the inverse grows slower.

The inverse of the function $y = ae^{kt}$ is $t = \frac{1}{k} \ln\left(\frac{y}{a}\right)$.

3. EXPONENTIAL GROWTH AND DECAY

We now have everything we need to explore exponential growth and decay. We will only be working with the base e exponential and logarithmic functions:

$$y = f(t) = ae^{kt} \quad \text{and its inverse} \quad t = \frac{1}{k} \ln\left(\frac{y}{a}\right).$$

This function can be used to model all the exponential growth and decay. We will explore its use through several applications.

Here are a few definitions and reminders.

- a is the initial amount ($f(0)$)
- k is the **relative growth rate**
- the function grows if $k > 0$
- the function decays if $k < 0$
- if the function grows at a rate of $p\%$, then $k = \ln(1 + r)$, where r is the growth rate expressed as a decimal
- the inverse of the function is used to find time and relative growth rate

Example 8. The population of Javelina in the Underfield Zoo was 20 in 1990. The population has a relative growth rate of 3.5%. Find the number of Javelina in the Underfield Zoo in 2020.

Solution: We are given $a = 20$ and $k = 0.035$. This gives us the growth function

$$f(t) = 20e^{0.035t}, \quad t \text{ in years.}$$

We are asked to evaluate this function when $t = 20$.

$$f(20) = 20e^{0.035 \cdot 20} \approx 40.$$

Example 9. The number of bacteria in a petrie dish grows exponentially. There are 4 bacteria in the dish when first observed and 12 hours later there are 8.

- (1) Find the model for the population.

Solution: We know that the basic model is $y = ae^{kt}$ and we are given $a = 4$. This time we are not given k but we need to find it. There is enough information. We have $y = 8$ when $t = 12$. Using the given information and the inverse function, we can find k .

$$\begin{aligned} 12 &= \frac{1}{k} \ln\left(\frac{8}{4}\right) \\ 12 &= \frac{0.69314718056}{k} \\ 12k &= 0.69314718056 \\ k &= 0.057762265047 \end{aligned}$$

The model is

$$f(t) = 4e^{0.057762t}.$$

- (2) Find the time that it takes for the population to reach 30.

Solution: Now we have k from the previous part, a is the same, and we know $y = 30$. Once again we will need the inverse to find t .

$$t = \frac{1}{0.054462} \ln\left(\frac{30}{4}\right) = 17.3123404907 \cdot 1.32175583998 \approx 34.9 \text{ hours}$$

- (3) What will the number of bacteria be in 2 days?

Solution: Here we need to calculate the number of hours in 2 days and then put that number into our model from part 1. Since 2 days is 24 hours

$$f(24) = 4e^{(0.0577622 \cdot 24)} \approx 16$$

Example 10. An example of decay is radioactive decay. Radioactive substances lose their radioactivity over time. This feature is often used to date organic objects. The *half-life* of a radioactive substance is the amount of time it takes for half of it to be gone. The radioactive isotope beryllium 11 decays at a very rapid rate. If we begin with 500 grams, the model for this decay is

$$A(t) = 500e^{-0.502t}, \quad t \text{ in seconds.}$$

- (1) How much of the sample is left after 5 seconds?

Solution:

$$a(t) = 500e^{-0.502*5} \approx 40.63 \text{ grams}$$

- (2) What is the half-life of beryllium 11?

Solution: Now we need to find the time at which there are 250 grams as that is half of our initial amount.

$$t = \frac{1}{-0.502} \ln\left(\frac{250}{500}\right) = \frac{1}{-0.502}(-0.69314718056) \approx 1.38 \text{ seconds}$$

Example 11. If interest is compounded continuously, the exponential function ae^{kt} gives us the amount of money at time t when k is the interest rate expressed as a decimal and t is in years. If we have \$2000 to deposit into an account that pays 6.3% compounded continuously, when will the account be worth \$100,000?

Solution: We have $a = 2000$, $k = 0.063$, and $y = 100000$, we are asked for t .

$$t = \frac{1}{0.063} \ln\left(\frac{100000}{2000}\right) = \frac{3.91202300543}{0.063} \approx 62.1 \text{ years}$$