RIGHT TRIANGLE TRIGONOMETRY

Objectives:
After completing this section, you should be able to do the following:
- Calculate the lengths of sides and angles of a right triangle using trigonometric ratios.
- Solve word problems involving right triangles and trigonometric ratios.

Vocabulary:
As you read, you should be looking for the following vocabulary words and their definitions:
- hypotenuse
- adjacent side
- opposite side

Formulas:
You should be looking for the following formulas as you read:
- ratio for $\sin(A)$ where $A$ is a non-right angle in right triangle
- ratio for $\cos(A)$ where $A$ is a non-right angle in right triangle
- ratio for $\tan(A)$ where $A$ is a non-right angle in right triangle

We will complete our study with a further study of right triangles. We will look at trigonometric value as defined by ratios of the sides of a right triangle.

In the figure above, you can see the sides of a right triangle labeled. The side labeled hypotenuse is always opposite the right angle of the right triangle. The names of the other two sides of the right triangle are determined by the angle that is being discusses. In our case, we will be discussing the sides in terms of the angle labeled $A$. The angle $A$ is formed by the hypotenuse of the right triangle and the side of the right triangle that
is called \textit{adjacent}. The \textit{adjacent} side will always make up part of the angle that is being discussed and not be the hypotenuse. The side of the right triangle that does not form part of angle $A$ is called the \textit{opposite} side. The \textit{opposite} side will never form part of the angle being discussed.

The trigonometric function values of a particular value can be as the ratio of a particular pair of sides of a right triangle containing an angle of that measure. We will look at three particular trigonometric rations.

\begin{center}
\begin{tabular}{c|c}
\hline
\textbf{Trigonometric Ratios} & \\
\hline
\text{sin}(A) = \frac{\text{opposite}}{\text{hypotenuse}} & \\
\hline
\text{cos}(A) = \frac{\text{adjacent}}{\text{hypotenuse}} & \\
\hline
\text{tan}(A) = \frac{\text{opposite}}{\text{adjacent}} & \\
\hline
\end{tabular}
\end{center}

\textit{sin} = \text{shortened form of sine function}
\textit{cos} = \text{shortened form of cosine function}
\textit{tan} = \text{shortened form of tangent function}
$A$ = \text{the angle value}

We will use these ratios to answer questions about triangles below and then we will go through a couple of application problems.

\textbf{Example 1:}

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (10,0) -- (10,10) -- cycle;
\draw (0,0) -- (60:10) node[midway,above] {$A$};
\draw (0,0) -- (30:10) node[midway,right] {$B$};
\draw (0,0) -- (0:10) node[midway,below] {$y$};
\draw (0,0) -- (60:10) node[midway,above] {$10$};
\end{tikzpicture}
\end{center}
Solution:
We are being asked to find values for \( x \), \( y \), and \( B \). We will do the angle \( B \) first.

**Angle \( B \):**
We can find the measure of angle \( B \) without using any trigonometric ratios. What we need to remember to find this value is that the sum of the three angles of a triangle will always add up to 180 degrees. It does not matter the size or shape of the triangle. The sum of the three angles will always be 180 degrees. We know that one angle is a right angle. Its measure is 90 degrees. The measure of the other angle is given to be 60 degrees. Thus we just need to calculate
\[
90 + 60 + B = 180
\]
\[
150 + B = 180
\]
\[
B = 30
\]
Thus the measure of angle \( B \) is 30 degrees.

**Side \( x \):**
We will now work to find the length of side \( x \). We need to start by determining which angle we are going to use for our problem. As in the past, it is best to use an angle that is given. Thus we will be using the angle labeled 60 degrees. Our next step is to determine which side \( x \) is relative to the angle labeled 60 degrees. Since the side labeled \( x \) is opposite the right angle, it is the hypotenuse.

There are two trigonometric ratios that include the hypotenuse. Thus, we need to determine which one to use. This will be determined by the other side of the triangle whose measure we know. This is the side labeled 10. Since this side is not one of the sides of the triangle that makes up the angle labeled 60 degrees, it is the opposite side. This means that we need to use the trigonometric ratio that has both the hypotenuse and the opposite side. That ratio is the sine ratio. We will plug into that equation and solve for \( x \).
Through out this solution, we have left sin(60) in this form. This saves us from needing to round until the end of the problem. sin(60) is just a number that at the end of the problem can be calculated by our calculator. Our answer is approximately $x = 11.54701$.

Side $y$:
We will finish by finding the length of side $y$. As in the previous part of the problem, it is best to use the angle labeled 60 degrees. Our next step is to determine which side $y$ is relative to the angle labeled 60 degrees. Since the side labeled $y$ is forms part of the angle labeled 60 degrees, it is the adjacent side. There are two trigonometric ratios that include the adjacent side. Since the other side that is given is the side labeled 10 and this side is the opposite side (see explanation above), we will need to use the trigonometric ratio that has both the opposite side and the adjacent side. That ratio is the tangent ratio. We will plug into that equation and solve for $y$.

$$
tan(A) = \frac{\text{opposite}}{\text{adjacent}}
$$

$$
tan(60) = \frac{10}{y}
$$

$$
tan(60) = \frac{10}{y}
$$

$$
y \cdot tan(60) = 1(10)
$$

$$
y = \frac{10}{tan(60)}
$$
Through out this solution, we have left $\tan(60)$ in this form. This saves us from needing to round until the end of the problem. $\tan(60)$ is just a number that at the end of the problem can be calculated by our calculator. Our answer is approximately $y = 5.773503$.

Example 2:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:

![Right Triangle Diagram]

Solution:

As we solve this problem, we will leave out the explanations of how we determine the names of the sides of the triangle.

Angle $A$:
As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

\[
90 + 45 + A = 180 \\
135 + A = 180 \\
A = 45
\]

Thus the measure of angle $A$ is 45 degrees.

Side $x$:
Side $x$ forms part of the angle that is labeled to be 45 degrees, thus this is the adjacent side. We are also given the measure of the side opposite the angle to be 3. Thus we will want to use the tangent ratio.
Our answer is $x = 3$.

We did not need to use trigonometric ratios to find $x$. We could have used the fact that our triangle has two angles that are equal. Such a triangle is an isosceles triangle. We should recall that the sides of an isosceles triangle opposite the equal angles are equal in length. Thus since one of the side was length 3, the side labeled $x$ is also of length 3.

Side $y$:

Side $y$ is opposite the right angle of the triangle and thus is the hypotenuse. We also have given opposite side to be 3. Thus to find $y$, we will need to use the sine ratio.

\[
\sin(45) = \frac{3}{y}
\]

\[
\sin(45) = \frac{3}{1} = \frac{3}{y}
\]

\[
y \sin(45) = 1(3)\\
y = \frac{3}{\sin(45)}
\]

Our answer is approximately $y = 4.24264$.

**Example 3:**

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:
Solution:

Angle $B$:
As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate
\[
\begin{align*}
90 + 53.4 + B &= 180 \\
143.4 + B &= 180 \\
B &= 36.6
\end{align*}
\]
Thus the measure of angle $B$ is 36.6 degrees.

Side $c$:
We will use the given angle labeled 53.4 degrees as our angle for solving this problem. Side $c$ forms part of the triangle that is opposite the right angle. Thus it is the hypotenuse. We are also given the measure of the side adjacent to the angle we are using to be 5.6. Thus we will want to use the cosine ratio.
\[
\begin{align*}
\cos(53.4) &= \frac{5.6}{c} \\
\cos(53.4) &= \frac{5.6}{c} \\
c \cdot \cos(53.4) &= 1(5.6) \\
c &= \frac{5.6}{\cos(53.4)}
\end{align*}
\]
Our answer is approximately $c = 9.39243$.

Side $a$:
We will use the given angle labeled 53.4 degrees as our angle for solving this problem. Side $a$ does not form the angle we are using. Thus it is the opposite side. We are also given the measure of the side adjacent to the angle we are using to be 5.6. Thus we will want to use the tangent ratio.
\[
\begin{align*}
\tan(53.4) &= \frac{a}{5.6} \\
\tan(53.4) &= \frac{a}{5.6} \\
5.6 \cdot \tan(53.4) &= 1(a) \\
5.6 \cdot \tan(53.4) &= a
\end{align*}
\]
Our answer is approximately $a = 7.54041$. 
Example 4:
Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:

\[ \begin{align*}
7.0 & \quad \text{B} \\
8.0 & \quad \text{A} \\
c & \quad \text{C}
\end{align*} \]

Solution:
In this problem, we are not given any angle to use. Instead we will need to change the labels of our sides as we solve each of the angles in turn. We will start by finding side \( c \). Since this is a right triangle, we can use the Pythagorean theorem to find the length of \( c \).

**Side \( c \):**
The legs \( (a \text{ and } b) \) are given to be 7.0 and 8.0. It does not matter which we label \( a \) and which we label \( b \).

\[ a^2 + b^2 = c^2 \]
\[ 7.0^2 + 8.0^2 = c^2 \]
\[ 49 + 64 = c^2 \]
\[ 113 = c^2 \]
\[ \sqrt{113} = c \]

The approximate length of side \( c \) is 10.63015.

**Angle \( A \):**
As we solve for angle \( A \), we need to label the sides whose measures are given relative to angle \( A \). The side labeled 8.0 forms part of the angle \( A \). Thus it is the adjacent side. The side labeled 7.0 does not form any part of the angle \( A \). Thus it is the opposite side. The trigonometric ratio that includes both the adjacent and opposite sides is the tangent ratio. We will fill in the information to that equation and solve for \( A \).
\[
\tan(A) = \frac{\text{opposite}}{\text{adjacent}}
\]
\[
\tan(A) = \frac{7.0}{8.0}
\]
\[
\tan(A) = 0.875
\]

We need to know how to solve for \( A \) in this equation. As in our section on exponential functions and their inverses, there is an inverse function (a function that undoes) for the tangent function. On the calculator it is labeled \( \tan^{-1} \). Thus we can finally solve for \( A \) by calculating
\[
\tan(A) = 0.875
\]
\[
A = \tan^{-1}(0.875)
\]

Thus the measure of angle \( A \) is approximately \( A \approx 41.18593 \) degrees.

Angle \( B \):

As we solve for angle \( B \), we need to relabel the sides whose measures are given relative to angle \( B \). The side labeled 7.0 forms part of the angle \( B \). Thus it is the adjacent side. The side labeled 8.0 does not form any part of the angle \( B \). Thus it is the opposite side. The trigonometric ratio that includes both the adjacent and opposite sides is the tangent ratio. We will fill in the information to that equation and solve for \( A \).

\[
\tan(A) = \frac{\text{opposite}}{\text{adjacent}}
\]
\[
\tan(A) = \frac{8.0}{7.0}
\]

We will once again use the inverse function of the tangent function. On the calculator it is labeled \( \tan^{-1} \). Thus we can finally solve for \( B \) by calculating
\[
\tan(B) = \frac{8.0}{7.0}
\]
\[
B = \tan^{-1}\left( \frac{8.0}{7.0} \right)
\]

Thus the measure of angle \( B \) is approximately \( B \approx 48.81407 \) degrees.
Example 5:
Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:

Solution:
Angle $A$:
As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

$$\begin{align*}
90 + 43.9 + A &= 180 \\
133.9 + A &= 180 \\
A &= 46.1
\end{align*}$$

Thus the measure of angle $A$ is 46.1 degrees.

Side $a$:
We will use the given angle labeled 43.9 degrees as our angle for solving this problem. Side $a$ forms part of the angle labeled 43.9 degrees. Thus it is the adjacent side. We are also given the measure of the hypotenuse to be .86. Thus we will want to use the cosine ratio.

$$\cos(43.9) = \frac{a}{.86}$$

Our answer is approximately $a = 0.619674$.

Side $b$:
We will use the given angle labeled 43.9 degrees as our angle for solving this problem. Side $b$ does not form the angle we are using. Thus it is the opposite side. We are also given the measure of the hypotenuse to be .86. Thus we will want to use the sine ratio.
\[
\sin(43.9) = \frac{b}{.86}
\]

\[
\sin(43.9) = \frac{b}{1}
\]

\[
.86 \cdot \sin(43.9) = 1(b)
\]

\[
.86 \cdot \sin(43.9) = b
\]

Our answer is approximately \(b = 0.596326\).

We will finish by looking at some application problems for our right triangle trigonometric ratios.

Example 6:
A support cable runs from the top of the telephone pole to a point on the ground 47.2 feet from its base. If the cable makes an angle of 28.7° with the ground, find (rounding to the nearest tenth of a foot)

a. the height of the pole
b. the length of the cable

Solution:
The picture above shows that we have a right triangle situation.

Part a:
The pole is opposite the angle of 28.7 degrees that is given. The other side of the triangle that we know to be 47.2 feet forms part of the angle of 28.7 degrees. Thus it is the adjacent side. We will be able to use the tangent ratio to solve this problem since it includes both the opposite and adjacent sides.
\[
\tan(A) = \frac{\text{opposite}}{\text{adjacent}}
\]

\[
\tan(28.7) = \frac{\text{pole}}{47.2}
\]

\[
\tan(28.7) = \frac{\text{pole}}{1}
\]

\[
47.2\tan(28.7) = 1(\text{pole})
\]

\[
47.2\tan(28.7) = \text{pole}
\]

Thus the pole is approximately 25.8 feet tall.

Part b:
The cable is opposite the right angle of triangle and thus is the hypotenuse. We still know that the adjacent side is 47.2 feet. We will be able to use the cosine ratio to solve this problem since it includes both the hypotenuse and adjacent sides.

\[
\cos(A) = \frac{\text{adjacent}}{\text{hypotenuse}}
\]

\[
\cos(28.7) = \frac{47.2}{\text{cable}}
\]

\[
\cos(28.7) = \frac{47.2}{1}
\]

\[
\text{cable} \cdot \cos(28.7) = 1(47.2)
\]

\[
\text{cable} = \frac{47.2}{\cos(28.7)}
\]

Thus the cable is approximately 53.8 feet tall.

Example 7:
You are hiking along a river and see a tall tree on the opposite bank. You measure the angle of elevation of the top of the tree and find it to be \(62.0^\circ\). You then walk 45 feet directly away from the tree and measure the angle of elevation. If the second measurement is \(48.5^\circ\), how tall is the tree? Round your answer to the nearest foot.
Solution:

This problem will require a little more algebra than the previous problems. We will start by looking at the two different triangles that we have and writing trigonometric ratios that include the tree for each of our triangles. We will start by looking at the bigger triangles (shown in red below).

In this red triangle, the tree is opposite the angle that is given to be 48.5 degrees. We are also given the length of 45 feet as part of the side that is adjacent to the angle given to be 48.5 degrees. Since we have part of this adjacent side, we are going to label the other part of the side to be $x$ feet. Thus the whole adjacent side is $45 + x$ feet long. Since we have the opposite and the adjacent sides, we can use the tangent ratio. For the red triangle, we have

$$\tan(x) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(48.5) = \frac{\text{tree}}{45 + x}$$

We will continue by looking at the smaller rectangle (shown in blue below).
In this blue triangle, the tree is opposite the angle that is given to be 62 degrees. Based on what we did for the red triangle, we know that the length of the side adjacent to the angle given to be 62 degrees is \( x \) feet long. Since we have the opposite and the adjacent sides, we can use the tangent ratio. For the red triangle, we have:

\[
\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}
\]

\[
\tan(62) = \frac{\text{tree}}{x}
\]

Now we are ready for the algebra. We have two equations:

\[
\tan(48.5) = \frac{\text{tree}}{45 + x}
\]

\[
\tan(62) = \frac{\text{tree}}{x}
\]

with two unknowns (variable). We can use the substitution method (solve one of the equations for one of the variable and then plug that in to the other equation) to determine the height of the tree. One way to go here is to solve the equation \( \tan(62) = \frac{\text{tree}}{x} \) for \( x \). This will give us:

\[
\tan(62) = \frac{\text{tree}}{x}
\]

\[
\frac{\tan(62)}{1} = \frac{\text{tree}}{x}
\]

\[
x \tan(62) = 1(\text{tree})
\]

\[
x = \frac{\text{tree}}{\tan(62)}
\]

We can now plug this expression for \( x \) into the equation:

\[
\tan(48.5) = \frac{\text{tree}}{45 + x}
\]

and solve for the height of the tree.
\[
\tan(48.5) = \frac{\text{tree}}{45 + x}
\]
\[
\tan(48.5) = \frac{\text{tree}}{45 + x}
\]
\[
(45 + x)\tan(48.5) = \text{tree}
\]
\[
45\tan(48.5) + x\tan(48.5) = \text{tree}
\]
\[
45\tan(48.5) + \left(\frac{\text{tree}}{\tan(62)}\right)\tan(48.5) = \text{tree}
\]
\[
45\tan(48.5) = \text{tree} - \left(\frac{\text{tree}}{\tan(62)}\right)\tan(48.5)
\]
\[
45\tan(48.5) = \text{tree}\left(1 - \left(\frac{1}{\tan(62)}\right)\tan(48.5)\right)
\]
\[
\frac{45\tan(48.5)}{\left(1 - \left(\frac{1}{\tan(62)}\right)\tan(48.5)\right)} = \text{tree}
\]

We now just need to plug this expression into our calculator to find out the height of the tree. The tree is approximately 127 feet tall.